Notes on Characteristic Classes

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Abstract

Characteristic classes are introduced. The index theorem associated with the dirac operator is presented.

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Characteristic classes are subsets of the (usually, de Rham) cohomology classes of the base space, and measure the non-triviality or twisting of a fibre bundle.

1 de Rham cohomology

Let M be a $m-$dimensional manifold, and $\Omega^r(M)$ the space of $r-$forms in M. The sequence induced by the exterior derivative [1]

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \ldots \to \Omega^{m-1}(M) \to \Omega^m(M) \to 0$$

is called the *de Rham* **complex**.

Since $d^2 \equiv d_{r+1}d_r = 0$, then $im\ d_r \subset ker\ d_{r+1}$. A **closed** r-form $\omega \in \Omega^r(M)$ is an element of $ker\ d_r$, *i.e.*, $dw = 0$. An **exact** r-form is an element of $im\ d_{r-1}$, *i.e.*, if there exists an $(r-1)-$form $\eta \in \Omega^{r-1}(M)$ such that $w = d\eta$. The rth *de Rham* **cohomology group** $H^r(M)$ is the quotient space of the of the set of closed r-forms, $ker\ d_r$, and of the set of exact r-forms, $im\ d_{r-1}$,

$$H^r(M) \equiv ker\ d_r\ /\ im\ d_{r-1}$$

I.e., two closed $r-$forms w_1, w_2 are identified in H^r if $w_1 - w_2 = d\eta$ for some $\eta \in \Omega^{r-1}(M)$.

2 Chern classes

Let (E, π, M, C^k, G) be a complex vector bundle, with gauge potential $\mathcal{A}$ and field strength $\mathcal{F}$ (naturally, with values in $\mathcal{L}(G)$).

The **total Chern class** is

$$c(\mathcal{F}) \equiv det\ (1 + \frac{i\mathcal{F}}{2\pi}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \ldots$$

where the individual (jth) **Chern classes**, $c_j(\mathcal{F}) \in \Omega^{2j}(M)$, are

$$\begin{aligned}
c_0(\mathcal{F}) &= 1 \\
c_1(\mathcal{F}) &= \tfrac{i}{2\pi}\ tr\ \mathcal{F} \\
c_2(\mathcal{F}) &= \tfrac{1}{2}(\tfrac{i}{2\pi})^2\ [\ tr\ \mathcal{F} \wedge tr\ \mathcal{F} - tr\ \mathcal{F} \wedge \mathcal{F}] \\
&= \tfrac{1}{2}[\ (tr\ \mathcal{A})^2\ -\ tr\ (\mathcal{A})^2] \\
c_3(\mathcal{F}) &= \tfrac{\pi}{6}(\tfrac{i}{2\pi})^3\ [2\ tr\ \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} - 3(tr\ \mathcal{F} \wedge \mathcal{F}) \wedge tr\ \mathcal{F}\ +\ tr\ \mathcal{F}\ \wedge tr\ \mathcal{F}\ \wedge tr\ \mathcal{F}] \\
&\vdots \\
c_k(\mathcal{F}) &= det\ A\ =\ (\tfrac{i}{2\pi})^k\ det\ \mathcal{F}
\end{aligned}$$

[1] Let us append a label to the exterior derivative, d_r, specifying the order r of the forms in which it acts; *i.e.*, d_r acts on elements of the space $\Omega^r(M)$

The series terminates at $c_k(\mathcal{F}) \;=\; det\, \frac{i\mathcal{F}}{2\pi}$, and $c_{j>k} = 0$. $c_j(\mathcal{F})$ is closed, thus $[c_j(\mathcal{F})] \in H^{2j}(M)$.

The **Pontrjagin class** is defined equivalently for the case of real vector bundles.

3 Chern character

The **total Chern character** is

$$ch(\mathcal{F}) \;\equiv\; tr\; exp\,(\frac{i\mathcal{F}}{2\pi}) \;=\; \sum_{j=1} \frac{1}{j!}\; tr\; (\frac{i\mathcal{F}}{2\pi})^j$$

The jth **Chern character** $ch_j(\mathcal{F})$ is

$$ch_j(\mathcal{F}) \equiv \frac{1}{j!}\; tr\; (\frac{i\mathcal{F}}{2\pi})^j$$

Each Chern character is expressed in terms of Chern classes as

$$\begin{gathered} ch_0(\mathcal{F}) = k \\ ch_1(\mathcal{F}) = c_1(\mathcal{F}) \\ ch_2(\mathcal{F}) = \tfrac{1}{2}[c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})] \\ \vdots \end{gathered}$$

k is the fibre dimension of the bundle.

For the case of an $SU(2)$ bundle over S^4, the total Chern character is

$$ch(\mathcal{F}) = 2 + \; tr\; (\frac{i\mathcal{F}}{2\pi}) + \frac{1}{2}\; tr\; (\frac{i\mathcal{F}}{2\pi})^2$$

The instanton number is given actually by

$$\int_{S^4} ch_2(\mathcal{F}) = \int_{S^4} \frac{1}{2}\; tr\; (\frac{i\mathcal{F}}{2\pi})^2$$

4 Chern-Simons form

An arbitrary 2-form characteristic class $P_j(\mathcal{F})$ being closed, it can be written locally [2] as an exact form by Poincaré's lemma,

$$P_j(\mathcal{F}) \;=\; dQ_{2j-1}(\mathcal{A}, \mathcal{F})$$

where $Q_{2j-1}(\mathcal{A}, \mathcal{F}) \;\in\; \mathcal{L}(G) \otimes \Omega^{2j-1}(M)$ is called the **Chern-Simons form** of $P_j(\mathcal{F})$.

[2] and not globally

If M is even-dimensional, $dim\ M = 2l$ and $\partial M \neq 0$, then Stoke's theorem implies

$$\int_M P_l(\mathcal{F}) = \int_M dQ_{m-1}(\mathcal{A},\mathcal{F}) = \int_{\partial M} Q_{m-1}(\mathcal{A},\mathcal{F})$$

Q_{m-1} is itself a characteristic class, describing the topology of the boundary ∂M.

Chern-Simons form of a Chern character

In particular, the Chern-Simons form of a Chern character is

$$Q_{2j-1}(\mathcal{A},\mathcal{F}) = \frac{1}{(j-1)!}\left(\frac{i}{2\pi}\right)^j \int_0^1 dt\ str(\mathcal{A},\mathcal{F}_t^{j-1})$$

where $\mathcal{F}_t = td\mathcal{F} + t(t-1)\mathcal{A}^2$ and str is the symmmetrized trace. Examples: [3]

$$Q_1(\mathcal{A},\mathcal{F}) = \frac{i}{2\pi} tr\mathcal{A}$$

$$Q_3(\mathcal{A},\mathcal{F}) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 tr[\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3]$$

$$Q_5(\mathcal{A},\mathcal{F}) = \frac{1}{6}\left(\frac{i}{2\pi}\right)^3 tr[\mathcal{A}(d\mathcal{A})^2 + \frac{3}{2}\mathcal{A}^3 d\mathcal{A} + \frac{3}{5}\mathcal{A}^5]$$

$$\vdots$$

For the $SU(2)$ gauge theory, the expression for $Q_3(\mathcal{A},\mathcal{F})$ can be used to find the component expression of

$$ch_2(\mathcal{F}) = d\ Q_3(\mathcal{A},\mathcal{F})$$

namely,

$$tr\ [\ \epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\ ] = \partial_\mu\ [\ 2\epsilon^{\mu\nu\rho\sigma}\ tr\ (\mathcal{A}_\nu\partial_\rho\mathcal{A}_\sigma + \frac{2}{3}\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma)]$$

Gauge transformation of Chern-Simons form

[3] $Q_5(\mathcal{A},\mathcal{F})$ appears in the formulation of the Wee-Zummino-Witten term

Let $\mathcal{A}_t,\ \mathcal{F}\ (t \in [0,1])$ be forms interpolating between the gauge field $\mathcal{A}$ and its gauge transformed $\mathcal{A}^g = g^{-1}(\mathcal{A}+d)g$,

$$\mathcal{A}_t = g^{-1}dg + tg^{-1}\mathcal{A}g, \qquad \mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2$$

The transformation formula for $Q_{2j+1}(\mathcal{A},\mathcal{F})$ is

$$Q_{2j+1}(\mathcal{A}^g,\mathcal{F}^g) = Q_{2j+1}(\mathcal{A},\mathcal{F}) + Q_{2j+1}(g^{-1}dg, 0) + d\alpha_{2m}(\mathcal{A}, g^{-1}dg)$$

where

$$\alpha_{2m}(\mathcal{A}, g^{-1}dg) = \int_0^1 dt\ l_t Q_{2j+1}(\mathcal{A}_t, \mathcal{F}_t)$$

l_t being a differential operator that acts on $\mathcal{A}_t, \mathcal{F}_t$ as $l_t\mathcal{A}_t = 0, \quad l_t\mathcal{F}_t = dt(\mathcal{A}_1 - \mathcal{A}_0)$.

5 Index theorem

Index theorems state relationships between the *analytic* properties of differential operators on fibre bundles and the *topological* properties of the fibre bundles themselves.

The index of an operator, determined by the zero-frequency solutions, is expressed in terms of the characteristic classes of the fibre bundles involved.

Differential operators — such as the Laplacian, the d'Alembertian, the Dirac operator — are regarded as maps of sections

$$D : \Gamma(M,E) \to \Gamma(M,F)$$

where $\Gamma(M,F),\ \Gamma(M,E)$ denote the set of sections on the base manifold M of vector bundles F, E.

The Dirac operator, in particular, is a map $\Gamma(M,E) \to \Gamma(M,E)$, E being a spin bundle over M.

If inner products are defined on the fibre manifolds E, F, it is then possible to define the adjoint of D,

$$D^\dagger : \Gamma(M,F) \to \Gamma(M,E)$$

The index of D is

$$ind\ D \ \equiv\ dim\ ker\ D\ -\ dim\ ker\ D^\dagger$$

where $ker\ D,\ \ ker\ D^\dagger$ are the sets of zero-eigenvectors of $D,\ D^\dagger$

$$\begin{aligned} ker\ D &\equiv \{\ s \in \Gamma(M,E) \mid D\ s\ =\ 0\ \} \\ ker\ D^\dagger &\equiv \{\ s \in \Gamma(M,F) \mid D^\dagger\ s\ =\ 0\ \} \end{aligned}$$

This analytical quantity is a topological invariant expressed in terms of the integral of an appropriate characteristic class over M.

5.1 The Atiya-Singer index theorem

Let M be a $2n$-dimensional Euclidean spacetime manifold, and D the Dirac operator with a gauge field $\mathcal{A}$

$$D[\mathcal{A}] \equiv \gamma^\mu(\partial_\mu + \mathcal{A}_\mu)$$

where γ^μ are the Euclidean Dirac gamma matrices.

Define the right-chirality Dirac operator $iD_R \equiv iDP_R$, where P_R is given in terms of γ_5 which is taken to be the product of the $2n$ Dirac gamma matrices.

The index of iD_R is

$$ind(iD_R)[\mathcal{A}] \;=\; dim\ ker\ (iD_R[\mathcal{A}]) \;-\; dim\ ker\ (iD_R[\mathcal{A}])^\dagger \;\equiv\; n_+ - n_-$$

with $n_\pm$ the number of eigenstates ϕ_0 of $iD[\mathcal{A}]$ with zero eigenvalue and chirality $\pm$, *i.e.*, $\gamma_5\phi_0 = \pm\phi_0$.

The **Atiya-Singer** index theorem states that [4]

$$ind(iD_R)[\mathcal{A}] \;=\; \int_M [ch(\mathcal{F})]_{vol} \;=\; \frac{1}{n!}(\frac{i}{2\pi})^n \int_M tr\ \mathcal{F}^n$$

[4]the subscript $_{vol}$ indicates that only the term proportional to the volume form of M contributes